Exercise Sheet #2

Course Instructor: Ethan Ackelsberg
Teaching Assistant: Felipe Hernández

P1. **(Problem 2.2.)** Prove that the extended real line $[-\infty, \infty]$ is homeomorphic to the closed unit interval $[0, 1]$.

P2. **(Problem 2.3.)** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $[-\infty, \infty]$, and let $c \in \mathbb{R}$. If $(x_n)_{n\in\mathbb{N}}$ converges to an extended real number, then the sequence $(cx_n)_{n\in\mathbb{N}}$ also converges, and

$$\lim_{n\to\infty} (cx_n) = c \cdot \lim_{n\to\infty} x_n. \tag{1}$$

P3. Let X, Y be two sets and $f : X \to Y$ a function.

(a) Prove that if $\mathcal{F}_Y \subseteq P(Y)$ is a σ-algebra, then $\mathcal{F}_X := \{f^{-1}(A) \mid A \in \mathcal{F}_Y\}$ is a σ-algebra.

(b) Prove that for $\mathcal{C} \subseteq P(Y)$, we have that $\sigma(f^{-1}(\mathcal{C})) = f^{-1}(\sigma(\mathcal{C}))$.

P4. Let $(X, \mathcal{T}, \mu)$ be a finite measure space, and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. Show that if $\mathcal{A}$ generates $\mathcal{T}$, then for every $B \in \mathcal{T}$ and for every $\epsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A\Delta B) \leq \epsilon$.

P5. Let X a set. We say that $\mathcal{S} \subseteq \mathcal{P}(X)$ is a semialgebra if

- $\emptyset, X \in \mathcal{S}$,
- if $A, B \in \mathcal{S}$ then $A \cap B \in \mathcal{S}$, and
- if $A, B \in \mathcal{S}$, then $A \setminus B = \bigsqcup_{i=1}^{n} C_i$ for some $C_i \in \mathcal{S}$.

Show that if S is a semialgebra, then

$$\mathcal{A}(S) = \left\{ \bigcup_{i=1}^{n} A_i \mid n \in \mathbb{N}, A_i \in \mathcal{S} \right\}.$$

Can we replace $\bigcup$ by $\bigsqcup$?